

CONVERGENCE OF A HYBRID SCHEME FOR THE ELLIPTIC MONGE-AMPÈRE EQUATION

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ABSTRACT. We prove the convergence of a hybrid discretization to the viscosity solution of the elliptic Monge-Ampère equation. The hybrid discretization uses a standard finite difference discretization in parts of the computational domain where the solution is expected to be smooth and a monotone scheme elsewhere. A motivation for the hybrid discretization is the lack of an appropriate Newton solver for the standard finite difference discretization on the whole domain.

1. INTRODUCTION

In this paper, we prove convergence to the viscosity solution of the elliptic Monge-Ampère equation of a hybrid discretization. The discretization we analyze is a variant of the hybrid method proposed by Froese and Oberman in [11]. The elliptic Monge-Ampère equation is a fully nonlinear equation, i.e. nonlinear in the highest order derivatives. Unless the domain is smooth and strictly convex and the data are smooth, the solution is not expected to be smooth. Nevertheless, numerical experiments [6, 2] indicate that the discrete equations obtained through standard finite difference discretizations have a discrete convex solution in the sense that a certain discrete Hessian is positive. The solution can be retrieved through appropriate iterative methods. But the convergence of the standard finite difference discretization in that sense to the viscosity solution remained obscure till [2] where a regularization approach is taken. However it is not known whether an appropriate Newton solver can be developed for the standard finite difference discretization. We note that the standard finite difference discretization is commonly used in science and engineering [15, 9, 8].

The hybrid discretization proposed in [11] uses a consistent, monotone and stable scheme in parts of the domain where the solution is not expected to be smooth and a standard discretization in parts of the domain where the solution is smooth. We will often refer to a consistent, monotone and stable scheme simply as a monotone scheme. The monotone discretization is known to converge to the viscosity solution when used on the whole domain with a convergent Newton's method solver [10]. As pointed out in [12] the convergence of the hybrid discretization introduced in [11] is still an open problem and results with the hybrid discretization of [11] are comparable with the ones obtained with the filtered approach in [12].

In this paper, we build on the recent advances in [2] on the analysis of the standard finite difference discretization. Combined with the classical framework for convergence of monotone schemes to viscosity solutions, we obtain convergence to the viscosity

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solution of our hybrid discretization. A fixed point argument is used to show existence of a discrete convex solution. The discretization we analyze differs from the one presented in [11] only by the choice of the standard finite discretization. One of the key arguments used in [2] is that, in the context of the standard finite difference discretization, the discrete Hessian, for a mesh function near a strictly convex smooth solution, is positive definite. We advocate the use of the compatible standard finite difference discretization introduced in [2] because it allows to transfer to the discrete level arguments for smooth solutions of the Monge-Ampère equation.

With a monotone discretization one can transfer to the discrete level arguments for viscosity solutions for partial differential equations. But it does not allow to give, in general, results for the standard finite difference discretization for smooth solutions. In fact, the quadratic convergence rate of the standard finite difference discretization for smooth solutions was only known as “formally second-order accurate” [6]. Moreover, the theory of Barles and Souganidis [5] cannot be applied directly to a hybrid discretization. For the latter and a standard finite difference discretization, pure numerical analysis techniques seem required. One of the main difficulties overcome in this paper is the choice of a suitable norm to measure the error resulting from a hybrid discretization. The impact of the results of this paper goes beyond the particular application considered. For example, the techniques used here may equally be applied to a hybrid scheme for the convex envelope presented in [19].

Numerical experiments reported in [11] indicate that Newton’s method can be applied to the nonlinear system resulting from a hybrid discretization. Although central difference approximations of the second order derivatives are used in [11], it is easily seen that the general methodology presented in this paper can be combined with new results on standard finite discretizations presented in [1] to yield the convergence of the original method used in [11]. For these reasons we do not give numerical results for the variant analyzed in this paper.

This paper is organized as follows. In the second section, we recall the notion of viscosity solution and present the hybrid discretization. We also define our notion of discrete convex function in the second section and the main notation of the paper. Existence and uniqueness of a discrete convex solution is given in the third section. In the fourth section we use the now classical arguments of [5] and new arguments for smooth solutions and the standard finite difference discretization of [2] to prove convergence to the viscosity solution of the hybrid discretization.

2. VISCOSITY SOLUTIONS OF THE ELLIPTIC MONGE-AMPÈRE EQUATION AND THE HYBRID DISCRETIZATION

For simplicity we consider a cuboidal domain $\Omega = (0, 1)^n \subset \mathbb{R}^n$. For given $f > 0$ continuous on $\overline{\Omega}$ and g continuous on $\partial\Omega$, with a convex extension $\tilde{g} \in C(\overline{\Omega})$, we consider the Monge-Ampère equation

$$(2.1) \quad \begin{aligned} \det D^2 u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega. \end{aligned}$$

Let $h > 0$ denote the mesh size. We assume without loss of generality that $1/h \in \mathbb{Z}$. Put

$$\begin{aligned}\mathbb{Z}_h &= \{x = (x_1, \dots, x_n)^T \in \mathbb{R}^n : x_i/h \in \mathbb{Z}\} \\ \Omega_0^h &= \Omega \cap \mathbb{Z}_h, \Omega^h = \overline{\Omega} \cap \mathbb{Z}_h, \partial\Omega^h = \partial\Omega \cap \mathbb{Z}_h = \Omega^h \setminus \Omega_0^h.\end{aligned}$$

For $x \in \mathbb{R}^n$, we denote the maximum norm of x by $|x| = \max_{i=1, \dots, n} |x_i|$. The norm $|\cdot|$ is extended canonically to matrices.

Let $\mathcal{M}(\Omega^h)$ denote the set of real valued functions defined on Ω^h , i.e. the set of mesh functions. For a subset T_h of Ω^h , and $v^h \in \mathcal{M}(\Omega^h)$ we define

$$|v^h|_{T_h} = \max_{x \in T_h} |v^h(x)|.$$

The norm $|\cdot|_{T_h}$ is extended canonically to matrix fields.

Let v be a continuous function on Ω and let $r_h(v)$ denote the unique element of $\mathcal{M}(\Omega^h)$ defined by

$$r_h(v)(x) = v(x), x \in \Omega^h.$$

We extend the operator r_h canonically to vector fields and matrix fields. For a function g defined on $\partial\Omega$, $r_h(g)$ defines the analogous restriction on $\partial\Omega^h$.

2.1. Viscosity solutions. A convex function $u \in C(\overline{\Omega})$ is a viscosity solution of (2.1) if $u = g$ on $\partial\Omega$ and for all $\phi \in C^2(\Omega)$ the following holds

- at each local maximum point x_0 of $u - \phi$, $f(x_0) \leq \det D^2\phi(x_0)$
- at each local minimum point x_0 of $u - \phi$, $f(x_0) \geq \det D^2\phi(x_0)$, if $D^2\phi(x_0) \geq 0$.

As explained in [16], the requirement $D^2\phi(x_0) \geq 0$ in the second condition above is natural at least in even dimensions. The space of test functions in the definition above can be restricted to the space of strictly convex quadratic polynomials [13, Remark 1.3.3].

An upper semi-continuous convex function u is said to be a viscosity sub solution of $\det D^2u(x) = f(x)$ if the first condition holds and a lower semi-continuous convex function is said to be a viscosity super solution when the second holds. A viscosity solution of (2.1) is a continuous function which satisfies the boundary condition and is both a viscosity sub solution and a viscosity super solution.

Note that the notion of viscosity solution is a pointwise notion. It is not very difficult to prove that if u is C^2 at x_0 , then u is a viscosity solution at the point x_0 of $\det D^2u = f$.

For further reference, we recall the comparison principle of sub and super solutions, [16, Theorem V. 2]. Let u and v be respectively sub and super solutions of $\det D^2u(x) = f(x)$ in Ω and put

$$u^* = \limsup_{y \rightarrow x, y \in \Omega} u(y) \text{ and } v_* = \liminf_{y \rightarrow x, y \in \Omega} v(y).$$

Then if $\sup_{x \in \partial\Omega} \max(u^*(x) - v_*(x), 0) = M$, then $u(x) - v(x) \leq M$ in Ω .

There are very few references which give an existence and uniqueness result for (2.1) in the degenerate case $f \geq 0$. In [16] it is required that one can find a sub solution and a super solution. The difficulty is that the Monge-Ampère equation is not often

studied in convex but not necessarily strictly convex domains. Thus we assume in addition that $f > 0$. Since $f \in C(\overline{\Omega})$ it follows that there exists a constant $c_0 > 0$ such that

$$f \geq c_0 > 0.$$

We also assume that g can be extended to a convex function $\tilde{g} \in C(\overline{\Omega})$. Then by [14, Theorem 1.1], (2.1) has a unique Aleksandrov solution. We can then use the equivalence of viscosity and Aleksandrov solutions [13, Propositions 1.3.4 and 1.7.1].

2.2. A reformulation of convexity. We recall that a function $\phi \in C^2(\Omega)$ is convex on Ω if the Hessian matrix $D^2\phi$ is positive semidefinite or $\lambda_1[\phi] \geq 0$ where $\lambda_1[\phi]$ denotes the smallest eigenvalue of $D^2\phi$. This notion was extended to continuous functions in [19]. See also the remarks on [21, p. 226]. A continuous function u is convex if and only if it is a viscosity solution of $-\lambda_1[u] \leq 0$ (or $\lambda_1[u] \geq 0$), that is, for all $\phi \in C^2(\Omega)$, whenever x_0 is a local minimum point of $u - \phi$, $-\lambda_1[\phi] \leq 0$ (resp. $\lambda_1[\phi] \geq 0$ for a local maximum). This can also be written $\max(-\lambda_1[u], 0) = 0$ in Ω , c.f. [19].

The Dirichlet problem for the Monge-Ampère equation (2.1) can then be written

$$(2.2) \quad \begin{aligned} \det D^2u - f &= 0 \text{ in } \Omega \\ \max(-\lambda_1[u], 0) &= 0 \text{ in } \Omega, \end{aligned}$$

with boundary condition $u = g$ on $\partial\Omega$. We write (2.2) as $F(u) = 0$ and note that the form of the equation is chosen to be consistent with the definition of ellipticity used for example in [16].

2.3. Standard finite difference discretizations. The version of the standard finite difference discretization we consider was first introduced in [2]. Let Ω_r be a bounded convex domain of \mathbb{R}^n . Put

$$\Omega_r^h = \overline{\Omega}_r \cap \mathbb{Z}_h.$$

Let $e^i, i = 1, \dots, n$ denote the i -th unit vector. We define first order difference operators acting on functions defined on \mathbb{Z}_h . For $x \in \mathbb{Z}_h$

$$\begin{aligned} \partial_+^i v^h(x) &:= \frac{v^h(x + he^i) - v^h(x)}{h} \\ \partial_-^i v^h(x) &:= \frac{v^h(x) - v^h(x - he^i)}{h}. \end{aligned}$$

We define discrete analogues of the gradient D_h and \overline{D}_h as:

$$\begin{aligned} D_h v^h &:= (\partial_+^i v^h)_{i=1, \dots, n} \\ \overline{D}_h v^h &:= (\partial_-^i v^h)_{i=1, \dots, n}. \end{aligned}$$

For $v^h = (v^{h,i})_{i=1, \dots, d}$, $v^{h,i} \in \mathcal{M}(\Omega^h)$ for all i , we define

$$\operatorname{div}_h v^h = \sum_{i=1}^d \partial_-^i v^{h,i},$$

and

$$\overline{D}_h v^h = (\partial_-^j v^{h,i})_{i,j=1, \dots, n}.$$

The discrete Hessian is defined as

$$\mathcal{H}_d(v^h) := \overline{D}_h D_h v^h.$$

Put

$$\begin{aligned}\Omega_{r,0}^h &= \{x \in \Omega_r^h, \mathcal{H}_d(v^h)(x) \text{ is defined for } v^h \in \mathcal{M}(\Omega^h)\} \\ \partial\Omega_r^h &= \Omega_r^h \setminus \Omega_{r,0}^h.\end{aligned}$$

For a matrix A , we recall that the cofactor matrix $\text{cof } A$ is defined by $(\text{cof } A)_{ij} = (-1)^{i+j} d(A)_i^j$ where $d(A)_i^j$ is the determinant of the matrix obtained from A by deleting the i th row and the j th column. For two matrices $A = (A_{ij})_{i,j=1,\dots,n}$ and $B = (B_{ij})_{i,j=1,\dots,n}$ we recall the Frobenius inner product $A : B = \sum_{i,j=1}^n A_{ij} B_{ij}$.

The discrete version of (2.1) on Ω_r takes the form

$$(2.3) \quad \frac{1}{n} \text{div}_h[(\text{cof sym } \mathcal{H}_d u^h)^T D_h u^h] = r_h(f) \text{ in } \Omega_{r,0}^h, u^h = r_h(\tilde{g}) \text{ on } \partial\Omega_r^h,$$

where for a matrix A , $\text{sym } A = (A + A^T)/2$ denotes the symmetric part of A .

We recall that the motivation of the above form of the discretization is to be able to transfer to the discrete level arguments for smooth solutions of the Monge-Ampère equation. For simplicity, we define for a mesh function v^h

$$(2.4) \quad M_r[v^h] = \frac{1}{n} \text{div}_h[(\text{cof sym } \mathcal{H}_d v^h)^T D_h v^h].$$

Let $\mathcal{M}(\Omega_r^h)$ denote the set of real valued functions defined on Ω_r^h . We define an inner product on $\mathcal{M}(\Omega_r^h)$ by

$$\langle v^h, w^h \rangle = h^n \sum_{x \in \Omega_{r,0}^h} v^h(x) w^h(x), v^h, w^h \in \mathcal{M}(\Omega_r^h),$$

and the following semi norms

$$\begin{aligned}||v^h||_{0,h} &= \sqrt{\langle v^h, v^h \rangle}, ||v^h||_{1,h} = \left(||v^h||_{0,h}^2 + \sum_{i=1}^n ||\partial_+^i v^h||_{0,h}^2 \right)^{\frac{1}{2}}, \\ |v^h|_{1,h} &= \left(\sum_{i=1}^n ||\partial_+^i v^h||_{0,h}^2 \right)^{\frac{1}{2}}, |v^h|_{2,\infty,h} = \max\{ \partial_-^j \partial_-^i v^h(x), x \in \Omega_{r,0}^h, i, j = 1, \dots, n \}.\end{aligned}$$

Put

$$H_0^1(\Omega^h) = \{v^h \in \mathcal{M}(\Omega^h), ||v^h||_{1,h} < \infty, v^h = 0 \text{ on } \partial\Omega^h\}.$$

We have

Lemma 2.1 (Discrete Poincaré's inequality). *There exists a constant $C_p > 0$ independent of h such that for $v^h \in H_0^1(\Omega^h)$,*

$$|v^h|_{1,h} \geq C_p ||v^h||_{0,h}.$$

The proof of the above result can be found in [20, Proposition 3.3].

2.4. Monotone schemes. Let us denote by $F_h(u^h) = F_h(u^h(x), u^h(y)|_{y \neq x})$ a discretization of $F(u)$. We recall the elements of the convergence theory of Barles and Souganidis [5] and how its conditions were met by the discretization introduced in [10].

The scheme $F_h(u^h) = 0$ is said to be monotone if for v^h and w^h in $\mathcal{M}(\Omega^h)$, $v^h(y) \geq w^h(y), y \neq x$ implies $F_h(v^h(x), v^h(y)|_{y \neq x}) \geq F_h(w^h, v^h(y)|_{y \neq x})$.

The scheme is said to be consistent if for all C^2 functions ϕ , and a sequence $x_h \rightarrow x \in \Omega$, $\lim_{h \rightarrow 0} F_h(r_h(\phi))(x_h) = F(\phi)(x)$.

Finally the scheme is said to be stable if $F_h(u^h) = 0$ has a solution u^h which is bounded independently of h .

It follows from [5] that a consistent, stable and monotone scheme has a solution u^h which converges locally uniformly to the unique viscosity solution of (2.2). Note that the convexity assumption on the exact solution is enforced through the definition of $F(u)$.

Next, we recall the notion of degenerate ellipticity of a scheme and that of proper scheme introduced in [18]. For $x \in \Omega^h$, let us denote by $N(x)$ a set of mesh points which are within a certain fixed distance from x . The choice of $N(x)$ introduces another discretization error, called directional error in [18]. Without loss of generality, we assume in this paper that $N(x) = \Omega_0^h$ for all x . We now assume that the discretization takes the form

$$F_h(u^h) = F_h(u^h(x), u^h(x) - u^h(y)|_{y \neq x, y \in N(x)}).$$

The scheme is said to be degenerate elliptic if it is nondecreasing in each of the variables $u^h(x)$ and $u^h(x) - u^h(y), y \in N(x), y \neq x$.

The scheme is proper if there is $\delta > 0$ such that for $x \in \Omega_0^h$ and for all $y \in \mathbb{R}^{N(x)}$, $x_0 \leq x_1$ implies $F_h(x_0, y) - F_h(x_1, y) \leq \delta(x_0 - x_1)$.

The scheme $F_h(u^h) = 0$ is Lipschitz continuous if there is $K > 0$ such that for all $x \in \Omega_0^h$ and $\alpha, \beta \in \mathbb{R}^{N(x)+1}$

$$|F_h(\alpha) - F_h(\beta)| \leq K|\alpha - \beta|_\infty.$$

It is not very difficult to prove that a degenerate elliptic scheme is monotone. Moreover for a scheme which is proper, degenerate elliptic and Lipschitz continuous, the equation $F_h(u^h) = 0$ has a unique solution to which converges the iteration

$$(2.5) \quad u_{k+1}^h = u_k^h - \nu F_h(u_k^h),$$

for ν sufficiently small [18, Theorem 7].

We recall the expression of the consistent, monotone and stable discretization of $\lambda_1[u]$ introduced in [19]. For simplicity we consider only wide stencils as the theoretical developments of this paper can be easily extended to discretizations with smaller stencils. We have

$$(2.6) \quad \lambda_1^h[u^h](x) = \min_{\alpha^h \in \mathbb{R}^n} \frac{u^h(x + \alpha^h) - 2u^h(x) + u^h(x - \alpha^h)}{|\alpha^h|^2},$$

where by $\alpha^h \in \mathbb{R}^n$ we mean vectors α^h for which the above expression is well defined for grid points.

We also recall the expression $M_s[u^h]$ of the discretization of $\det D^2u$ used in [10]. For $x \in \Omega_0^h$ we denote by $W_h(x)$ the set of orthogonal bases of \mathbb{R}^n such that for $(\alpha_1, \dots, \alpha_n) \in W_h(x)$ $x \pm \alpha_i \in \Omega^h, \forall i$. We have

$$(2.7) \quad M_s[u^h] = \inf_{(\alpha_1, \dots, \alpha_n) \in W_h(x)} \prod_{i=1}^n \frac{u^h(x + \alpha_i) - 2u^h(x) + u^h(x - \alpha_i)}{|\alpha_i|^2}.$$

Now, the scheme introduced in [10] can be shown to be stable using the contraction mapping principle used to show convergence of the iteration (2.5). See Section 4 for a similar situation. It is proper as it can be written $F_h(u^h) = F_h(u^h(x) - u^h(y)|_{y \neq x, y \in N(x)})$, degenerate elliptic and consistent [10]. Moreover it is also Lipschitz continuous, see for example [18].

The monotone discretization of (2.2) can then be written

$$(2.8) \quad \begin{aligned} M_s[u^h](x) - r_h(f)(x) &= 0, x \in \Omega^h \\ \max(-\lambda_1^h[u^h](x), 0) &= 0, x \in \Omega^h \\ u^h(x) &= r_h(g)(x) \text{ on } \partial\Omega^h. \end{aligned}$$

It follows from the above discussion that (2.8) has a unique solution which converges locally uniformly to the unique solution of (2.2).

As with [10], the first two equations of (2.8) are combined in a single equation. Recall that $x^+ = \max(x, 0)$ and define

$$M_s^+[u^h] = \inf_{(\alpha_1, \dots, \alpha_n) \in W_h(x)} \prod_{i=1}^n \max\left(\frac{u^h(x + \alpha_i) - 2u^h(x) + u^h(x - \alpha_i)}{|\alpha_i|^2}, 0\right).$$

Then (2.8) can be written

$$(2.9) \quad \begin{aligned} M_s^+[u^h](x) - r_h(f)(x) &= 0, x \in \Omega^h \\ u^h(x) &= r_h(g)(x) \text{ on } \partial\Omega^h. \end{aligned}$$

2.5. The hybrid discretization.

Definition 2.2. We call a point $x \in \Omega$ a regular point if the solution u of (2.1) is C^2 in a neighborhood of x . A point which is not a regular point is called a singular point.

The above definition is natural if one considers the one dimensional Monge-Ampère equation $-u''(x) = f$ and a standard finite difference approximation. In particular, at a regular point x , by a Taylor series expansion,

$$(2.10) \quad \lim_{h \rightarrow 0} \max_{i,j=1,\dots,n} |\partial^2 v(x) / (\partial x_i \partial x_j) - \partial_-^j \partial_+^i (r_h v)(x)| = 0.$$

Next, for $v \in C^4(\Omega)$, and $x \in \Omega$

$$(2.11) \quad \max_{i,j=1,\dots,n} |\partial^2 v(x) / (\partial x_i \partial x_j) - \partial_-^j \partial_+^i (r_h v)(x)| \leq Ch^2 |v|_{4,\Omega},$$

where for an integer j , $|v|_{j,\Omega} = \sup_{|\beta|=j} \sup_{\Omega} |D^\beta v(x)|$ for a multi-index β .

Let Ω_r denote an open subset of Ω such that at every point x of Ω_r the exact solution u is C^2 in a neighborhood of x . Using the notation of section 2.3 we define

$$\Omega_s^h = \Omega_0^h \setminus \Omega_{r,0}^h.$$

Given that an adaptive finite difference procedure for the discretization of the Monge-Ampère equation has yet to be developed, as with [11], one can take a conservative approach and include a priori in Ω_s^h , points where either $f(x)$ is not Hölder continuous, $f(x)$ is too small or $f(x)$ is too large. It is very likely that the approximation will deteriorate at points which are close to boundary points where $\partial\Omega$ is not C^3 or strictly convex and points close to boundary points where $g(x)$ cannot be extended to a C^3 function. They may be included in Ω_s^h as well. The motivation to consider these points as singular points come from the regularity theory of the Monge-Ampère equation. See for example Theorem 1.1 in [22].

Definition 2.3. *By a discrete convex function, we mean a mesh function v^h such that*

$$(2.12) \quad \begin{aligned} \lambda_1^h[v^h] &\geq 0 \text{ in } \Omega_s^h \\ \mathcal{H}_d v^h(x) &\geq 0 \text{ in } \Omega_{r,0}^h. \end{aligned}$$

Strictly discrete convex functions are defined analogously.

We note that a discrete convex function in the sense of the above definition is not necessarily convex on Ω_0^h . See [19] for the case $\Omega_s^h = \Omega_0^h$ and [17] for a counterexample showing that the discrete Hessian $\mathcal{H}_d v^h$ can be positive without the mesh function v^h being convex in the usual sense. The minor abuse of terminology we make is justified by Theorem 4.4 below which says in particular that the uniform limit of mesh functions which satisfy (2.12) and solve the discrete Monge-Ampère equation (2.15) below, is convex.

For a subset T_h of Ω^h , we denote by $\mathcal{C}^h(T_h)$ the cone of discrete convex functions on T_h and by $\mathcal{C}_0^h(T_h)$ the cone of strictly discrete convex functions on T_h . We define on Ω_0^h for a mesh function v^h , $F_h(v^h)$ by

$$(2.13) \quad \begin{aligned} F_h(v^h)(x) &= M_s^+[v^h](x) - r_h(f)(x), x \in \Omega_s^h \\ F_h(v^h)(x) &= M_r^+[v^h](x) - r_h(f)(x), x \in \Omega_{r,0}^h. \end{aligned}$$

Put $\mathcal{C}^h = \mathcal{C}^h(\Omega_0^h)$ and define a norm $|\cdot|_h$ on $\mathcal{M}(\Omega^h)$ by

$$(2.14) \quad |v^h|_h = \max\left\{ |v^h|_{\Omega_s^h}, \frac{h^{-\frac{n}{2}}}{C_p} |v^h|_{1,h} \right\},$$

where we denote by C_p the constant in the Poincaré's inequality for the domain Ω_r and we recall that the semi-norm $|\cdot|_{1,h}$ takes only into account mesh points in $\Omega_{r,0}^h$. To verify that the above formula defines a norm, one uses the observation that $\partial\Omega_{r,0}^h \subset \Omega_s^h$ and Lemma 2.1.

The hybrid discretization of (2.2) can then be written: find $u^h \in \mathcal{C}^h$

$$(2.15) \quad F_h(u^h)(x) = 0 \text{ in } \Omega_0^h, u^h(x) = r_h(g)(x) \text{ on } \partial\Omega^h.$$

In [11], the authors use a weight function to write the hybrid discretization as a combination of the monotone scheme and the standard finite difference discretization. We omit it in this paper as it plays no role in our analysis.

3. EXISTENCE AND UNIQUENESS OF A DISCRETE CONVEX SOLUTION

We first show that the problem (2.15) has a unique local solution. We define a linear operator L_h on $\mathcal{M}(\Omega^h)$ as

$$\begin{aligned} L_h v^h(x) &= v^h(x), x \in \Omega_s^h \\ L_h v^h(x) &= \Delta_h^{-1} v^h(x), x \in \Omega_{r,0}^h, \end{aligned}$$

and where the operator $\Delta_h = \operatorname{div}_h D_h v^h$ is considered only on $\Omega_{r,0}^h$, i.e. if $w^h = \Delta_h^{-1} v^h$, then

$$\Delta_h w^h = v^h \text{ on } \Omega_{r,0}^h \text{ and } w^h = 0 \text{ on } \partial\Omega_r^h.$$

Consider the ball

$$B_\rho(r_h(u)) = \{ v^h \in \mathcal{M}(\Omega^h), |v^h - r_h(u)|_{1,h} \leq \rho, v^h = r_h(g) \text{ on } \partial\Omega^h, v^h = r_h(u) \text{ on } \partial\Omega_{r,0}^h \}.$$

Recall that the semi norm $|\cdot|_{1,h}$ takes only into account mesh points in $\Omega_{r,0}^h$. As with [2] and [3], it will be necessary to use a “rescaling argument”, i.e. solve a rescaled equation in the ball $\alpha B_\rho(r_h(u))$ for

$$\alpha = h^{\frac{3+\frac{n}{2}}{n-1}}.$$

Next we define the mapping

$$\begin{aligned} (3.1) \quad S : \mathcal{M}(\Omega^h) &\rightarrow \mathcal{M}(\Omega^h) \\ S(\alpha v^h)(x) &= \alpha v^h(x) - \nu_x \alpha^n L_h F_h(v^h)(x) \\ \nu_x &= \nu_1, x \in \Omega_s^h \\ \nu_x &= \nu_2, x \in \Omega_{r,0}^h, \end{aligned}$$

for $\nu_1, \nu_2 > 0$. Our goal is to show that for ρ, h sufficiently small and an appropriate ν , S has a fixed point in $B_\rho(r_h(u))$. We have

Lemma 3.1. *There exists a positive constant $a_1 < 1$ such that for all $v^h, w^h \in \mathcal{M}(\Omega^h)$, we have*

$$|S(\alpha v^h) - S(\alpha w^h)|_{\Omega_s^h} \leq a_1 |\alpha v^h - \alpha w^h|_{\Omega_0^h},$$

for $C_0 \leq \nu_1 \leq C_1$ where C_0 and C_1 are positive constants.

Proof. The proof essentially follows the one for monotone schemes on the whole domain given in [18, Theorem 7]. \square

We have

$$|\alpha v^h - \alpha w^h|_{\Omega_0^h} = \max\{ |\alpha v^h - \alpha w^h|_{\Omega_s^h}, |\alpha v^h - \alpha w^h|_{\Omega_{r,0}^h} \}.$$

Since $|v_h|_{\Omega_{r,0}^h}^2 \leq \sum_{x \in \Omega_{r,0}^h} |v_h(x)|^2$, we obtain

$$(3.2) \quad \begin{aligned} |v_h|_{\Omega_{r,0}^h} &\leq h^{-\frac{n}{2}} \|v_h\|_{0,h} \\ &\leq \frac{h^{-\frac{n}{2}}}{C_p} |v_h|_{1,h}. \end{aligned}$$

It follows from the results of [2] that the operator $S|_{\Omega_{r,0}^h}$ is a strict contraction in a rescaled ball and maps the rescaled ball in itself. In other words, there exists $\nu_2, \rho > 0$ and a constant $a_2, 0 < a_2 < 1$, which depends on h such that for $v^h, w^h \in B_\rho(r_h(u))$,

$$|S(\alpha v^h) - S(\alpha w^h)|_{1,h} \leq a_2 |\alpha v^h - \alpha w^h|_{1,h},$$

for $\rho = Ch^{2+n/2}$ and h sufficiently small.

Moreover, for $v^h \in B_\rho(r_h(u))$ we have

$$|S(\alpha v^h) - \alpha r_h(u)|_{1,h} \leq \alpha \rho.$$

Using the definition of the hybrid norm (2.14) on $\mathcal{M}(\Omega^h)$, we easily obtain

$$|S(\alpha v^h) - S(\alpha w^h)|_h \leq \max\{a_1, a_2\} |\alpha v^h - \alpha w^h|_h.$$

The existence of a unique local solution to (2.15) then follows from the Banach fixed point theorem. For the choice of an initial guess, we recall that the nine-point finite difference approximation to the solution of a Poisson equation produces an approximation of order 4.

4. CONVERGENCE TO THE VISCOSITY SOLUTION OF THE HYBRID DISCRETIZATION

We first prove the stability of the hybrid discretization (2.15). Then we prove that the half-relaxed limits

$$\begin{aligned} u^*(x) &= \limsup_{y \rightarrow x, h \rightarrow 0} u^h(y) = \lim_{\delta \rightarrow 0} \sup \{ u^h(y), y \in \Omega_0^h, |y - x| \leq \delta, 0 < h \leq \delta \} \\ u_*(x) &= \liminf_{y \rightarrow x, h \rightarrow 0} u^h(y) = \lim_{\delta \rightarrow 0} \inf \{ u^h(y), y \in \Omega_0^h, |y - x| \leq \delta, 0 < h \leq \delta \}, \end{aligned}$$

are respectively sub and super solutions of (2.2).

4.1. Stability on the set of singular points. Let $x \in \Omega_s^h$ and let ν be a vector such that both $x + \nu h$ and $x - \nu h$ are in Ω_0^h . For $\phi(x) = 1/2|x|^2$, the second order directional derivative

$$D_{\nu\nu}\phi = \frac{1}{|\nu|^2 h^2} (\phi(x + \nu h) - 2\phi(x) + \phi(x - \nu h)),$$

is a constant equal to 1. It follows that $M_s^+(r_h(\phi))$ is a constant. And since we assumed that f is continuous, f is bounded on $\bar{\Omega}$ and hence by (2.13), $|F_h(r_h\phi)| = | -M_s^+(r_h(\phi))(x) + r_h(f)(x) |$ is bounded by a constant which does not depend on h and x . Since u^h is a fixed point of S , by (3.1) and the boundedness of the domain

$$\begin{aligned} |u^h|_{\Omega_s^h} &= |S(u^h)|_{\Omega_s^h} \leq |S(u^h) - S(r_h(\phi))|_{\Omega_s^h} + |S(r_h(\phi))|_{\Omega_s^h} \\ &\leq a_1 |u^h - r_h(\phi)|_{\Omega_0^h} + |r_h(\phi)|_{\Omega_s^h} + \nu_1 |F_h(r_h(\phi))|_{\Omega_s^h} \\ &\leq a_1 |u^h - r_h(\phi)|_{\Omega_0^h} + C, \end{aligned}$$

for a constant C . But

$$|u^h - r_h(\phi)|_{\Omega_0^h} = \max\{|u^h - r_h(\phi)|_{\Omega_s^h}, |u^h - r_h(\phi)|_{\Omega_{r,0}^h}\}.$$

Since $0 < a_1 < 1$, by triangular inequality, we have for some constant C

$$(4.1) \quad \text{either } |u^h|_{\Omega_s^h} \leq C \quad \text{or} \quad |u^h|_{\Omega_s^h} \leq |u^h|_{\Omega_{r,0}^h} + C.$$

4.2. Stability on the set of regular points. Since $\rho = Ch^{2+n/2}$, we have by (3.2) and assuming without loss of generality that $h \leq 1$

$$(4.2) \quad |u^h - r_h(u)|_{\Omega_{r,0}^h} \leq Ch^{-\frac{n}{2}} |u^h - r_h(u)|_{1,h} \leq Ch^2 \leq C.$$

Thus for h sufficiently small,

$$(4.3) \quad |u^h|_{\Omega_{r,0}^h} \leq |u^h - r_h(u)|_{\Omega_{r,0}^h} + |r_h(u)|_{\Omega_{r,0}^h} \leq C,$$

since by definition $u \in C(\overline{\Omega})$.

Inequalities (4.1) and (4.3) allow us to state the following theorem

Theorem 4.1. *There is a constant $C > 0$ independent of h such that for h sufficiently small, the solution u^h of (2.15) satisfies $|u^h|_{\Omega_0^h} \leq C$.*

4.3. Sub and super solution property of the half-relaxed limits. Theorem 4.1 implies that the half-relaxed limits are well defined. We have

Theorem 4.2. *The upper half-relaxed limit u^* is a viscosity sub solution of $\det D^2u(x) = f(x)$ and the lower half-relaxed limit u_* is a viscosity super solution of $\det D^2u(x) = f(x)$ at every point of $\Omega \setminus \Omega_r$. In addition, they are both viscosity solutions of $-\lambda_1[u](x) \leq 0$ at every point of $\Omega \setminus \Omega_r$.*

Proof. The result follows from the results of [5] and the stability, consistency and monotonicity of the scheme used in the "singular" part of the domain. For the convenience of the reader, we give a proof following [7].

We show that u_* is a viscosity super solution of $\det D^2u(x) = f(x)$ and a viscosity solution of $-\lambda_1[u](x) \leq 0$ at every point of $\Omega \setminus \Omega_r$. The corresponding result for u^* is proved similarly.

It follows from the definitions that u_* is lower semi-continuous. Let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ with $D^2\phi(x_0) \geq 0$ such that $u_* - \phi$ has a local minimum at x_0 with $(u_* - \phi)(x_0) = 0$. Without loss of generality, we may assume that x_0 is a strict local minimum.

Let B_0 denote a closed ball contained in Ω and containing x_0 in its interior. We let x_n be a sequence in B_0 such that $x_n \rightarrow x_0$ and $u^{h_n}(x_n) \rightarrow u_*(x_0)$ and let x'_n be defined by

$$c_n := (u^{h_n} - \phi)(x'_n) = \min_{B_0} u^{h_n} - \phi.$$

Since the sequence x'_n is bounded, it converges to some x_1 after possibly passing to a subsequence. Since $(u^{h_n} - \phi)(x'_n) \leq (u^{h_n} - \phi)(x_n)$ we have

$$(u_* - \phi)(x_0) = \lim_{n \rightarrow \infty} (u^{h_n} - \phi)(x_n) \geq \liminf_{n \rightarrow \infty} (u^{h_n} - \phi)(x'_n) \geq (u_* - \phi)(x_1).$$

Since x_0 is a strict minimizer of the difference $u_* - \phi$, we conclude that $x_0 = x_1$ and $c_n \rightarrow 0$ as $n \rightarrow \infty$.

By definition

$$u^h(x) \geq \phi(x) + c_n, \forall x \in B_0,$$

and thus, by the monotonicity of the scheme

$$0 = F^h(u^h(x_0), u^h(y)|_{y \neq x_0}) \geq F^h(u^h(x_0), (\phi(y) + c_n)|_{y \neq x_0}),$$

which gives by the consistency of the scheme $-\det D^2\phi(x_0) + f(x_0) \leq 0$ and $-\lambda_1[\phi](x_0) \leq 0$. \square

For the behavior at regular points, we have

Theorem 4.3. *At every regular point $x \in \Omega$,*

$$u_*(x) = u^*(x) = u(x).$$

And thus u_ and u^* are viscosity solutions of (2.2) at $x \in \Omega_r$.*

Proof. By (4.2), u^h converges to u uniformly on compact subsets of Ω_r . The result then follows since u is C^2 at x . \square

We close this section by stating the main result of this paper

Theorem 4.4. *The solution u^h of (2.15) converges uniformly on compact subsets to the unique solution of (2.2).*

Proof. First using the definitions we have $u_* \leq u^*$. Second, by the comparison principle (recalled in section 2.1) and Theorems 4.2 and 4.3, we have $u_* \geq u^*$. Hence $u_* = u^*$ is the unique viscosity solution of (2.2). By uniqueness of the viscosity solution $u_* = u^* = u$ and hence u^h converges uniformly on compact subsets to u by [4, Lemma 1.9 p. 290]. \square

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